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| The Trashcan (Software and Media Publication, NFP) Presents |
| Applying Terrestrial Gravitational Motion in 2D Settings |
| A Physics Supplement for Application to Software, Including a Primer or Review of Basic Calculus and Newtonian Kinematics, Plus TerrestrialGravitation, a C++ Library |

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| Ray Arias  10 March 2018 to 29 April 2018 |

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**I humbly dedicate this supplemental paper and the accompanying software library to my loving wife, Maria A. Arias, for whose ceaseless work and assistance I will always be grateful.**

What is the first derivative of a cow? Prime rib!

—Numerous joke sites on the Internet

Falling in love is not at all the most stupid thing that people do, but gravitation cannot be held responsible for it.

—Albert Einstein

**Introduction**

You need to make a game, simulation, or other software in 2D and you need to account for the effect of gravity upon the objects in it? It’s not a problem. You just need to learn (or brush up on) your **terrestrial kinematics**. This is a basic Newtonian description (as opposed to Relativistic or Quantum and all that fancy newer stuff) of how gravity and motion work here on Earth’s surface (as opposed to in Earth’s center or in outer space or inside the sun, etc.).

As a note, there are other forces that can affect the path of physical objects in motion other than gravity. There is friction, both static and kinetic, with gaseous air particles, but it is very small for most objects, except when dealing with winds of great velocity, as in tornados and hurricanes, or taking into account objects that can be greatly affected by wind resistance or can take advantage of aerodynamic lift, such as gliders, airplanes, and parachutes. The physics of wind resistance and aerodynamic lift are extremely complicated and are not covered in this short supplemental paper.

**Alternate Forms of 2D Notation**

I would like to explain that there are a number of ways to notate the same 2-dimensional point on a Cartesian (rectangular) plane, both mathematically and geometrically. For example, the same point *P* can be notated as coordinates (*a*, *b*), or as a complex number *a* + *bi*, or as the geometric vector *a****î*** + *b****ĵ*** (***î*** and ***ĵ*** are unit vectors in the positive *x* and *y* directions respectively to express a geometric vector using components).[[1]](#footnote-1)

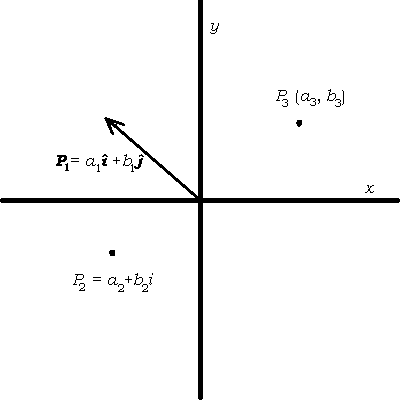
Comparison can only be done as to ascertain whether or not two points, such as *P*1 and *P*2, are equal or not, and *not* as to whether one is greater than or less than the other. Because each of the two points has two different values, neither one can be said to be greater than or less than the other. However, both points are equal, *if and only if*, both *a* values are equal *and* both *b* values are equal. These alternate systems of notation give rise to a number of operations that can be done to two or more points, such as finding equality, as well as addition and subtraction:

Figure 1: Example of points on a Cartesian plane, where *P*1 is a geometric vector, *P*2 is a complex number, and *P*3 is a pair of coordinates.

Although, operations as simple as addition, subtraction, and finding equality can be done using notation from any of these systems, others such as complex multiplication, complex division, and geometric vector dot product only have definitions specific to their notation system:

There is also a geometric vector cross product. However, since this cross product would introduce a third dimension and this paper only covers two-dimensional math, this will not be discussed in this paper.

Further, the distance from the origin to a point, also conceived as the absolute value of a complex number, or the magnitude of a geometric vector, can be figured using the Pythagorean Theorem and adding the squares of the coordinates of a point, of the real and imaginary numbers in a complex number, or of the unit vector coefficients of a geometric vector, and then taking the square root of the result. In each equation below, the absolute value bars (**|***x***|**) indicate the distance (in the case of the point *P* with coordinates (*a*, *b*)), absolute value (in the case of the complex number *q* with real number component *a* and imaginary number component *b*), or magnitude (in the case of the vector with ***î*** unit vector coefficient *a* and ***ĵ*** unit coefficient *b*) of the item between them.

Although, comparatively speaking, calculating the distance/absolute value/magnitude of a point/complex number/geometric vector is relatively simple, determining its angle from the horizontal right semiaxis, the positive *x*-axis, is a bit more complicated. This is due to the angular values by every inverse trigonometric function (also known as an “arc function”) repeating with every ½ rotation of the full circle. This, in turn, is due to the given values resulting from each standard trigonometric function doing the same with its values. Therefore, for every possible resulting value of any given standard trigonometric function, there are two possible angular values in the full 2π-radian, or 360°, rotation that correspond to this original value.

Nonetheless, because we already know the coordinates/real and imaginary components/unit vector coefficients, we can know what quadrant it is in (or which axis it is on, if applicable, or if it is the origin). Using this information in conjunction with the result of an inverse trigonometric function, such as the arctangent, an exact angle on the rotational circle can be calculated.

For the sake of simplicity, let us consider a point *P* with coordinates (*a*, *b*). If *P* is in Quadrant I (also known as the First Quadrant), above the *x*-axis and to the right of the *y*-axis, both *a* and *b* would be positive. But if *P* is in Quadrant II (also known as the Second Quadrant), it would still be above the *x*-axis, but this time to the left of the *y*-axis, *a* would be negative and *b* would be positive. Further, if *P* is in Quadrant III (also known as the Third Quadrant), it would now be both below the x-axis **and** to the left of the *y*-axis, and both *a* and *b* would be negative. Finally, if *P* is in Quadrant IV (also known as the Fourth Quadrant), it still be below the *x*-axis, but now to the right of the *y*-axis, and therefore, *a* would now be positive, but *b* would still be negative.

An integer function can be created using this information that gives 1 if P is in the First Quadrant, 2 if P is in the Second, 3 if it is in the Third, or 4 if in the Fourth. However, there are a few places on the Cartesian plane that don’t fall in any of the four Quadrants. These locations are along the x-axis, along the y-axis, and on the origin (which can be considered as being along both axes or neither). If the function we wish to create, which we can mathematically define as *quadrant*(*a* ϵ ℝ, *b* ϵ ℝ) ϵ ℤ | 1 ≤ *quadrant* ≤ 4 (or, “quadrant is a function of a, member of real numbers, and b, member of real numbers, which function value is a member of integers, such that quadrant is equal to or between 1 and 4”), we can define the *quadrant*(*a*, *b*) function to be *quadrant*(0, 0) = 0 when *P* is the origin, and then give this function negative values when *P* lies somewhere else on one of the axes. Let’s allow *P* on the positive *x*-axis yield *quadrant*(*a*, 0) = –1 (*a* > 0), on the positive *y*-axis give *quadrant*(0, *b*) = –2 (*b* > 0), on the negative *x*-axis make for *quadrant*(*a*, 0) = –3 (*a* < 0), and on the negative *y*-axis result in *quadrant*(0, *b*) = –4 (*b* < 0). We must now expand the range of the *quadrant*(*a*, *b*) function to be –4 ≤ *quadrant* ≤ 4 (or, *quadrant* is equal to or between -4 and 4). We can now create a complete table of every possible value for our *quadrant*(*a*, *b*) function as follows:

|  |  |
| --- | --- |
| **Location of point *P*** | **Value of *quadrant*(*a*, *b*)** |
| Negative *y*-axis (*a* = 0, *b* < 0) | –4 |
| Negative *x*-axis (*a* < 0, *b* = 0) | –3 |
| Positive *y*-axis (*a* = 0, *b* > 0) | –2 |
| Positive *x*-axis (*a* > 0, *b* = 0) | –1 |
| Origin (*a* = 0, *b* = 0) | 0 |
| Quadrant I (*a* > 0, *b* > 0) | 1 |
| Quardant II (*a* < 0, *b* > 0) | 2 |
| Quadrant III (*a* < 0, *b* < 0) | 3 |
| Quadrant IV (*a* > 0, *b* < 0) | 4 |

As mentioned above, once we can identify the Quadrant *P* is in, we can use this information to know which of the two angles compatible with the arctangent function is the correct one. Basically, we obtain the slope of the distance line by dividing *b* by *a*. Next, we use this slope-quotient and put it into the arctangent function, which customarily yields an angle (in radians) -π/2 < *φ* < π/2. If we require the resultant angle to be 0 ≤ *θ* < 2π, then when *quadrant*(*a*, *b*) = 4, we can add 2π to the angle yielded by the arctangent function. Since a point in Quadrant II would yield a negative slope-quotient *b*/*a* just like one in Quadrant IV, when *quadrant*(*a*, *b*) yields 2, we need to add π to this angle to obtain the correct one, and since a point in Quadrant III gives a positive slope-quotient, just like one in Quadrant I, when *quadrant*(*a*, *b*) yields 3, we also need to add π to the angle as well. Finally, for 0 and all the negative values yielded for *quadrant*(a, b), the appropriate angle is already known. For *P* at the origin (*quadrant*(0, 0) = 0), there is no valid angle as the distance is also 0. However, where a value is required, as in a software function, 0 would do just as good as any other. As for *P* on the positive *x*-axis, *θ* = 0; for *P* on the positive *y*-axis, *θ* = π/2; for *P* on the negative *x*-axis, *θ* = π; and for *P* on the negative *y*-axis, *θ* = 3π/2. The following is a mathematical summary of how the final angle is obtained, with *φ* being the angle yielded by the arctangent function and *θ* being the final desired angle.

Here are four examples of points from which we will obtain a distance from the origin and an angle from the right horizontal.

**Some Basic Calculus**

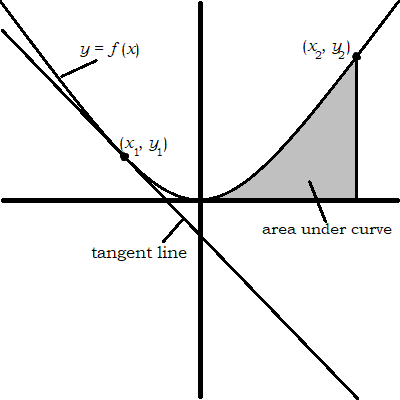
Before we go on, let us cover some other basics. If you don’t know calculus, that’s fine, but you will have to understand a few of its most essential concepts. Given a function *y* = *f*(*x*) (“f of x”), its **derivative** (“d y d x equals f prime of x”) is the rate of change of the function (or *y*) with respect to its parameter value (*x*), or more fundamentally, the slope of the function at the point (*x*, *y*). It should be noted here that the differentials (*dy* and *dx*) written as a fraction do in no way signify any actual rational fraction, it’s just one of a number of convenient ways to notate the derivative of a function. The derivative of a derivative is known as the **second derivative** of a function and is notated (“d squared y d x squared equals f double prime of x” or “…f second of x”). Further, the area between the *y*-axis and the curve made by a function, adding when the curve is above the *y*-axis and subtracting when the curve is below it, from 0 to *x* for *x* > 0, or *x* to 0 for *x* < 0 gives you its **integral**. Taking the integral of a derivative of a function will give you back the original function, particularly if you know any constants (such as C) that were part of the original function that got obliterated by the derivative. Here are four simple examples of how this occurs, the last of which shows a second derivative and its reintegration:

Figure 3: The slope of the tangent line gives the derivative of *f* (*x*) at (*x*1, *y*1) and the area under the curve gives the integral of *f* (*x*) at  
(*x*2, *y*2).

Notice that the derivative of any constant is always zero. (*f*(*x*) = *a*, *f’*(*x*) = 0) Also, the derivative of a linear function of *x* is just the coefficient of *x* as a constant. (*f*(*x*) = *ax*, *f’*(*x*) = *a*) Conversely, the integral of any constant is this constant multiplied by *x* (plus another constant C). Further, when taking the derivative of a quadratic function, the coefficient of *x*2 is multiplied by 2 and *x*2 becomes *x*. (*f*(*x*) = *ax*2, *f’*(*x*) = 2*ax*) Also, just as before the coefficient of *x* stands alone as a constant and is added to the previous term. (*f*(*x*) = *ax*2 + *bx*, *f’*(*x*) = 2*ax* + *b*) Reversing this process, in integrating a term with *x*, the coefficient is divided by 2 and *x* becomes *x*2. Just in **differentiation** (taking the derivative), terms also add together during **integration**.

**The Calculus of Kinematics**

When examining kinematics, we do not observe *y* with respect to *x*, by rather *x* and *y* each with respect to time (*t*). So an object’s position is actually *x*(*t*) and *y*(*t*), but we generally still use the conventional notation *x* and *y*. But the time parameter becomes apparent when we take the position’s derivative to obtain velocity, such as . The dot on the x (pronounced “x dot” or “x dot of t”) is the same as the prime sign above, except the derivative is taken with respect to *t*, as opposed to another variable.

The **position** (*x*, for horizontal position or *y*, for vertical position) of a physical object is its placement in relation to other objects or a coordinate system, such as the Cartesian plane. When the object is in motion, its **velocity** (*vx*, for horizontal velocity or *vy*, for vertical velocity) is the rate of change in its position with respect to time (*t*). This rate of change is expressed as either of the derivatives or . When this object’s velocity changes, the rate of this change is its **acceleration** (*ax* or *ay*), also known as or .

Just as the rates of change can be arrived at by taking derivatives, given a rate of change, the original kinematic attribute can be arrived at by taking an integral. So taking the integral of the acceleration of an object will give you back its velocity, assuming you know its initial velocity, and taking the integral of the velocity will give you back its position, assuming you know its initial position.

Terrestrial kinematics entails **horizontal kinematics**, which as you may guess is a description of horizontal motion, and **vertical kinematics**, a description of vertical motion. Here is each one in detail.

**Horizontal Kinematics**

Since any forces pushing or pulling upon any object in the horizontal while it is flying through the air are negligible, a solid object being thrown, catapulted, or otherwise projected into the air, can generally be considered as having neither any thrust added to, nor any resistance taken away from, its initial motion. Hence, the horizontal speed is practically constant; there is no horizontal component of acceleration to take into account. Consequently, the horizontal position only changes by adding the same constant as a coefficient of time to the initial position when animating in 2D software.

Notice how the initial kinematic attributes (*x*(0) and *vx*(0)) are constants and act the same way C does in integrals of functions of *y* with respect to *x*, like those in the introduction.

**Vertical Kinematics**

The vertical motion of objects flying through the air, for instance after being catapulted or thrown, is similar to that of an object, such as a pinball,, loaded on top of a coiled spring and then propelled by allowing the spring to uncoil. This object would then be shot upward by the force of the uncoiling spring. This upward force would immediately be counteracted by the downward force of terrestrial gravity. Once the object is out of contact with the spring, the object would continue upward with whatever velocity was given to it initially by the force of the spring. This upward velocity would reduce until it reaches zero as gravitation would be the only force affecting the object after that point. Additionally, after reaching its vertex (the highest point in its vertical path), the object would then gain a downward velocity from the continued gravitation affecting it. This velocity would be small at first and then grow to its maximum just before the object either reloads the spring, recoiling it in the process, or it hits the ground. Generally, a vertically propelled object returns with a downward velocity of the same magnitude, at the same height that it started, as the initial upward velocity that it was originally driven to by the upward force. For example, if a ball was thrown straight up at 10 miles/hour, by the time it finishes going up, gets to its vertex, and goes down, when it reaches the same height it was thrown up at, it should be going straight down at pretty much (if not *exactly*) 10 mi/hr.

After this point, if an object with downward velocity is allowed to make contact with the ground, depending upon what the ground is made of and what the object is made of (as well as the shape of the object and which part makes contact with the ground), the object will bounce back up at some fractional upward velocity of the downward velocity it came down with. For a rubber ball on a hard surface, this fraction is very large, near one, for a less bouncy object on a plush surface, this fraction is very small, near zero. This fraction is known as the coefficient of restitution (CoR, or *epsilon* ε). Each subsequent bounce results in CoR being reapplied until the y component of velocity of the object is close enough to zero as to be negligible.

In physics, a **force** gives rise to the acceleration of an object. A force is calculated using Isaac Newton’s equation *F* = *ma*, where *m* is the mass of the object and *a* is its acceleration. Terrestrial gravitation is one such force and the equation for it is a special case of Newton’s general force equation. The terrestrial force of gravity upon any object’s mass is *G*⊕,which is equal to *mg*, the mass of the object (*m*) multiplied by the acceleration given to it by the gravitation between the object and the Earth (*g*). This acceleration is expressed either in US Customary Units as 32.1740 feet/sec2 or in the International System of Units (SI) as 9.80665 meters/sec2. These are averages for mid-latitudes at average altitude and actual measurements for *g* can vary according to altitude from sea level as well as latitude on the Earth’s surface.

If these facts are applied separately to each of the axes, when plotted on a graph using sample initial velocities for each direction, the following is obtained:

If both axes are applied jointly, this graph is obtained:

**Conclusion**

You may be wondering what all this math and physics means for creating games, simulations, and other software. Well, I understand some people reading this may be having a hard time with math formulas, particularly those involving calculus, but basically, here is the basic breakdown. Since, on the horizontal axis, the velocity is going to be constant, you can assign a constant velocity

const double vx = 10.0;

x = x0 + (vx \* t);

(where vx is horizontal velocity, x is current horizontal position, x0 is initial horizontal position, and t is time) or neglect it entirely as you can just hard code the constant directly when you are changing the x-coordinate (x = x0 + (10 \* t);). For the vertical velocity, you will need a variable though. It is your choice, however, if you want to keep a constant, such as,

const double g = 9.8;

vy = vy0 – (g \* t);

y = y0 + (vy \* t);

(where g is the gravitational acceleration constant, vy is the current vertical velocity, vy0 is the initial vertical velocity, y is the current vertical position, and y0 is the initial vertical position) or hard code when you change your vertical velocity.

vy = vy0 – (9.8 \* t);

y = y0 + (vy \* t);

Nevertheless, keep in mind that 9.8 (m/s)/s—or whichever number you select for *g*—is a change that occurs every whole second, so if you are changing your vertical velocity 100 times per second, you will need to divide this number by 100. Thus, you should use either something like

const double g = 9.8;

vy = vy0 – ((g / 100) \* (t / 100);

or,

const double g100 = 0.98;

t100 = t / 100;

vy = vy0 – (g100 \* t100);

or even,

const double g = 9.8;

dt = 100;

vy = vy0 – ((g / dt) \* (t / dt));

Whichever works for you, when your object finally flies across the screen, you should get a similar shape to the second graph (concave down parabola). If not, you need to double check your math. Good luck!

**Appendix: Reference for the C++ Library TerrestrialGravitation**

I hope you find bundled with this document a software library I made in C++. This is to help you create objects that behave as if they were under the influence of terrestrial gravitation as well as in order to instruct you as to how to code such objects and the behavior underlying them appropriately. I call it, simply enough, TerrestrialGravitation.

By the way, I have decided that anyone who wants to use this commercially and proprietarily can either afford other software that would effectively do what this library already does, or can afford to effectively recode it from scratch by reverse engineering it, which they are more than welcome to do. This is why this library is released under harder CopyLeft license of GNU GPL, v3, rather than the lesser so LGPL.

These classes have constructors as fully defined as possible, from default constructors with no parameters, to fully defined constructors with every variable member defined as a parameter, to copy constructors, to assignment operators, to even destructors.

**class Cartesian**

The library starts with the class Cartesian. These are just objects that have integer x and y coordinates (int), as well as double precision floating point coordinates (double), designated as xx and yy. I made two sets of coordinates so that switching between doubles and ints would not be any excessive burden. For scientific precision, doubles are best used. However, for games and graphic simulations that require pixel coordinates, ints are much faster and convenient to use. Therefore, the Cartesian class has coordinates of both types.

Due to this, I inserted three functions to ease using objects that come out of this class. First is the Boolean function bool concordance(), which requires no parameters. This checks to see if the values of x and xx are within less than an integer unit’s difference equal. If this test passes, it checks for the same level of equality between y and yy. If this test passes, TRUE is returned, otherwise the result is FALSE. Next, is the void function concordanceInt(). Basically, all this does is force concordance, or agreement, between the ints and the doubles. In the case of concordanceInt(), this is accomplished by assigning the doubles to the values of the ints (casting them as doubles in the process, of course). Finally, the void function concordanceDouble() assigns the values of the doubles (casted as ints, and therefore, truncated) to the ints.

Notwithstanding the assignment operator (operator=(…)), there is also an equality operator (operator==(…)) and an in equality operator (operator!=(…)) that will evaluate to true and false respectively if two Cartesian objects are equal and unequal, and *vice versa* if not. I have also placed an addition operator (operator+(…)), as well as a subtraction operator (operator-(…)), in order to facilitate the addition and subtraction of coordinates of different Cartesian objects. There are also a multiplication operator (operator\*(…)), as well as a division operator (operator/(…)), in order to carry out complex multiplication and complex division. These operators treat the *x*-coordinates of both Cartesian objects as the real parts of two complex numbers and the *y*-coordinates as the imaginary parts and calculate the complex product or complex quotient according to the following formulas (where *c*1 is the first Cartesian object, *c*2 is the second Cartesian object, *p* is the complex product, and *q* is the complex quotient):

There are functions for finding distance from the Cartesian object to the origin, or between two Cartesian objects. This can be given as a double or an int, according to if you use distanceInt(…) or distanceDouble(…). Computing distance is done by using the Pythagorean theorem: *a*² + *b*² = *c*², or in this case, , where ***distance*** is negative if either *x* or *y*, but not both, is negative and positive if both *x* and *y* are positive or both are negative.

There are a pair of functions quadrant() and quadrant(double ww, double zz) that each return an int for the quadrant of the Cartesian object or the point given by (ww, zz). The values returned are 1 for Quadrant I (*x* positive, *y* positive), 2 for Quadrant II (*x* negative, *y* positive), 3 for Quadrant III (*x* negative, *y* negative), 4 for Quadrant IV (*x* positive, *y* negative), -1 for positive *x*-axis, -2 for positive *y*-axis, -3 for negative *x*-axis, -4 for negative *y*-axis, or 0 for the Origin (0, 0).

There are also functions that return the angle pointing away from the origin or the other Cartesian object. The angleInt(…)functions return degrees and the angleDouble(…)ones return radians. However, if you use angleInt(…), the doubles will be concordanceInt()ed with the ints, and if you use angleDouble(…), vice versa. Angles are calculated using the arctangent of *y* / *x*, or in C code: double thetaRad = atan(yy / xx); the Int version carries out this same calculation but multiplies the result by the constant 180/π, or again in C: int thetaDeg = (int) ((180.0 / PI) \* thetaRad); However, due to the limits of the arctan function, we must find the quadrant or axis the object is on, as you will see. Here are the formulas, where *q* is the integer result of the quadrant(…) function, *φ* is a working angle and *θ* is the final angle returned (A → B, in this context, means if A is true then do B):

The arctan function only normally gives back an angle of . And to further complicate things, as the angle gets closer to –π/2 or π/2, the closer *y*/*x* gets to negative or positive infinity. Therefore, using arctan alone to find an angle is not very effective without first knowing which quadrant the point is in or which axis it is on. So these functions make use of the quadrant(…) functions above. If the point is on the positive *x*-axis or in Quadrant I, the raw arctan result is returned. If the point is on the positive *y*-axis, π/2, or 90°, is returned. If the point is in Quadrant II, π, or 180°, is added to the angle. If the point is on the negative *x*-axis, π, or 180°, is returned. If the point is in Quadrant III, π, or 180°, is added to the angle. If the point is on the negative *y*-axis, 3π/2, or 270°, is returned. Finally, if the point is in Quadrant IV, in order for all angles returned to be positive in the range of , or , 2π, or 360°, is added to the angle. By the way, if the point is on the Origin, a default angle of 0, or 0°, is returned, however, it should be noted that such a point would have no valid angle.

There are functions for reading and assigning either or both of the ints x and/or y or either or both of the doubles xx and/or yy. After each assignment, one of the concordance assignment functions is executed so that agreement is maintained between ints and doubles.

There are also functions for inverting (giving the negative of) ints x and/or y or doubles xx and/or yy as well as any int or double passed as a parameter. This may be useful in certain contexts, such as drawing graphics where most raster screens put the Origin (0, 0) at the top left and the x-coordinate increases to the right while the y-coordinate increases downward.

There is an atOrigin() function that assigns everything to zero (referred to on Cartesian graphs as the Origin, or (0, 0)). In order to facilitate assignment to another Cartesian object, atOrigin() returns the result as well as assigning it to the current object (\*this).

Lastly, there is a function that will return the geometric vector dot product of the current Cartesian object and another Cartesian object given as a parameter. It does this by computing the following formula, where carteA is the current object and carteB is the object given as a parameter:

The class Cartesian, as you will see, is a fully accessible encapsulated class of class InertBody. All members are public, so they are all freely accessible. However, two Cartesian objects (position and velocity) will be used in class InertBody. Also, all members of class Cartesian are accessible by way of functions (for instance, int X() and void X(int a) to read and assign the Cartesian member int x respectively, etc.). There are also functions to write members jointly (void XY(int a, int b) and void XXYY(double aa, double bb)).

*Variable Members:* ints: x, y, doubles: xx, yy.

*Constructors:* Cartesian(void): (***default constructor***) assigns 0 to x and y and 0.0 to xx and yy.

Cartesian(int a, b): assigns a to x and b to y and executes concordanceInt() to make the doubles agree with the ints.

Cartesian(double aa, bb): assigns aa to xx and bb to yy and executes concordanceDouble() to make the ints agree with the doubles.

Cartesian(const Cartesian &cartecopy): (***copy constructor***) takes each member of a given Cartesian object and assigns its value to each member of the current object.

*Operators:* ***Assignment*** (Cartesian operator=(const Cartesian &carteassignment)): Same as ***copy constructor***, but checks for self-assignment, for which case nothing happens, and at the end the current object (\*this) is returned.

***Negation*** (Cartesian operator-(const Cartesian &cartenegative)): Returns the negative of a Cartesian object by assigning double –cartenegative.xx to double xx, assigning double –cartenegative.yy to double yy, and executing concordanceDouble().

***Addition*** (Cartesian operator+(const Cartesian &carte1, const Cartesian &carte2)): Adds the coordinates of two Cartesian objects and returns a third whose coordinates are the sum of the other two.

***Subtraction*** (Cartesian operator–(const Cartesian &carte1, const Cartesian &carte2)): Subtracts the coordinates of a second Cartesian object from the coordinates of the first one and returns a third object whose coordinates are the difference of the other two.

***Multiplication*** (Cartesian operator\*(const Cartesian &carte1, const Cartesian &carte2)): (***complex product***) Returns the complex product of two Cartesian objects by assigning (double carte1.xx \* double carte2.xx) – (double carte1.yy \* double carte2.yy) to double xx, assigning (double carte1.xx \* double carte2.yy) + (double carte1.yy \* double carte2.xx) to double yy, and executing concordanceDouble().

***Division*** (Cartesian operator/(const Cartesian &carte1, const Cartesian &carte2)): (***complex quotient***) Returns the complex quotient of two Cartesian objects by creating doubles a, b, c, d, qr, and qi, and making the following assignments:

a = carte1.xx; b = carte1.yy;

c = carte2.xx; d = carte2.yy;

qr = ((a \* c) + (b \* d)) / ((c \* c) + (d \* d));

qi = ((b \* c) – (a \* d)) / ((c \* c) + (d \* d));

Then qr is assigned to double xx, qi is assigned to double yy, and finally concordanceDouble() is executed.

***Equality***(bool operator==(const Cartesian &carteEqual1, const Cartesian &carteEqual2)): assigns false to a new bool equality and tests carteEqual1.concordance() and carteEqual2.concordance(). If both are true, double carteEqual1.xx is compared to double carteEqual2.xx to see if they are equal. If this result is true, double carteEqual1.yy is compared to double carteEqual2.yy to see if they are equal. If this result is true, true is assigned to equality. If any of the above tests fail, false will remain assigned to equality. Finally, equality is returned.

***Inequality***(bool operator!=(const Cartesian &carteUnequal1, const Cartesian &carteUnequal2)): assigns false to a new bool inequality and tests carteUnequal1.concordance() and then carteUnequal2.concordance(). If either is false, true is assigned to inequality. If both of these are true, double carteUnequal1.xx is compared to double carteUnequal2.xx to see if they are unequal. If so, true is assigned to inequality. Otherwise, double carteUnequal1.yy is compared to double carteUnequal2.yy to see if they are unequal. If so, true is assigned to inequality. Otherwise, if all these tests fail, false remains assigned to inequality. Finally, inequality is returned.

*Destructor:* ~Cartesian(void): deletes current Cartesian object.

*Functions for reading the current position:*

int X(void): returns the current object’s value for int x.

int Y(void): returns the current object’s value for int y.

double XX(void): returns the current object’s value for double xx.

double YY(void): returns the current object’s value for double yy.

Cartesian Position(void): returns a Cartesian object whose coordinates are identical to the current object’s coordinates.

*Functions for setting the current position:*

void X(int a): assigns a to int x, executes concordanceInt().

void Y(int b): assigns b to int y, executes concordanceInt().

void XY(int a, int b): assigns a to int x, assigns b to int y, executes concordanceInt().

void XX(double aa): assigns aa to double xx, executes concordanceDouble().

void YY(double bb): assigns bb to double yy, executes concordanceDouble().

void XXYY(double aa, double bb): assigns aa to double xx, assigns bb to double yy, executes concordanceDouble().

void Position(Cartesian pos): assigns pos’s coordinates to current object’s coordinates by way of assigning pos.xx to xx, assigning pos.yy to yy, and executing concordanceDouble().

*Functions that invert (negate) coordinates or numbers:*

void invertX(void): transforms x and xx to –x and –xx.

void invertY(void): transforms y and yy to –y and –yy.

void invertXY(void): transforms x, xx, y, and yy to –x, –xx, –y, and –yy.

int invert(int a): returns –a.

double invert(double aa): returns –aa.

*Functions that compute distance*:

int distanceInt(void): calculates distance from Cartesian object to origin (0, 0) to nearest integer unit and returns result.

double distanceDouble(void): calculates distance from Cartesian object to origin (0, 0) and returns result.

double distance(void): same as distanceDouble().

int distanceInt(int a, int b): calculates distance between from current Cartesian object’s coordinate to coordinates given by (a, b) to the nearest integer unit and returns the result.

int distance(int a, int b): same as distanceInt(int a, int b).

double distanceDouble(double aa double bb): calculates distance from current Cartesian object’s coordinates to coordinates given by (aa, bb) and returns the result.

double distance(double aa, double bb): same as distanceDouble(double aa, double bb).

int distanceInt(Cartesian carte): calculates the distance between the current Cartesian object and another Cartesian object given by carte to the nearest integer unit and returns the result.

double distanceDouble(Cartesian carte): calculates the distance between the current Cartesian object and another Cartesian object given by carte and returns the result.

double distance(Cartesian carte): same as distanceDouble(Cartesian carte).

*Functions for finding angles:*

int angleInt(void): computes the angle from the current Cartesian object to the origin (0, 0) in degrees to the nearest integer unit and returns the result.

double angleDouble(void): computes the angle from the current Cartesian object to the origin (0, 0) in radians and returns the result.

double angle(void): same as angleDouble(void).

int angleInt(int w, int z): computes the angle in degrees from the current Cartesian object to coordinates given by (w, z) to the nearest integer unit and returns the result.

int angle(int w, int z): same as angleInt(int w, int z).

double angleDouble(double ww, double zz): computes the angle in radians from the current Cartesian object to coordinates given by (ww, zz) and returns the result.

double angle(double ww, double zz): same as angleDouble(double ww, double zz).

int angleInt(Cartesian carte): calculates angle in degrees from current Cartesian object to another Cartesian object given by carte to the nearest integer unit and returns the result.

double angleDouble(Cartesian carte): calculates angle in radians from current Cartesian object to another Cartesian object given by carte and returns the result.

double angle(Cartesian carte): same as angleDouble(Cartesian carte).

*Functions for finding the quadrant point is in or axis point is on:*

int quadrant(void): returns an int -4 ≤ q ≤ 4 indicating where the Cartesian object is located. Value returned according to this table:

|  |  |
| --- | --- |
| **Location of Cartesian object** | **Value Returned** |
| Negative *y*-axis (*x* = 0, *y* < 0) | -4 |
| Negative *x*-axis (*x* < 0, *y* = 0) | -3 |
| Positive *y*-axis (*x* = 0, *y* > 0) | -2 |
| Positive *x*-axis (*x* > 0, *y* = 0) | -1 |
| Origin (*x* = 0, *y* = 0) | 0 |
| Quadrant I (*x* > 0, *y* > 0) | 1 |
| Quardant II (*x* < 0, *y* > 0) | 2 |
| Quadrant III (*x* < 0, *y* < 0) | 3 |
| Quadrant IV (*x* > 0, *y* < 0) | 4 |

int quadrant(double ww, double zz): same as quadrant(), except returns q according to location of (ww, zz) instead of location of object.

*Functions for checking and forcing concordance of* int *and* double *coordinate sets:*

bool concordance(void): checks to see if (x, y) and ((int) round(xx), (int) round(yy)) agree to the nearest integer unit. Returns TRUE if they do and FALSE if they don’t.

void concordanceInt(void): forces concordance, or agreement, between double and int coordinates, doubles are assigned double casted values of ints.

void concordanceDouble(void): forces concordance, or agreement, between double and int coordinates, ints assigned int casted values of doubles after the double round(double) function from <cmath> is executed.

*Miscellaneous Function:*

Cartesian atOrigin(void): current Cartesian object is assigned to be at the origin (0, 0) and (0.0, 0.0) and is returned to facilitate assignment.

double DotProduct(const Cartesian &carte): Returns the geometric dot product of current Cartesian object and carte by creating doubles mag1, mag2, ang, and dot and making the following assignments:

mag1 = distanceDouble();

mag2 = carte.distanceDouble();

ang = angleDouble(carte);

dot = mag1 \* mag2 \* cos(ang);

Finally, dot is returned.

**class InertBody**

The class InertBody is an encapsulator class of class Cartesian in that class InertBody makes use of class Cartesian by creating two objects from it, Cartesian position and Cartesian velocity, that are members of this class. There is also a protected member, bool \_at\_rest, that every function that sets Cartesian velocity tests to be (0, 0) by way of declaring a new Cartesian Zero and then executing its atOrigin() function and then comparing velocity to Zero. This class only applies position and velocity, but neither any gravitation, nor any other force, hence the name of this class.

*Constant defined by precompiler:* \_DEFAULT\_TIME is 30.

*Variable Members:* Cartesian objects: position, velocity, initialPosition.

**Protected** *Variable Members:* bool \_at\_rest, ints: \_n, \_t.

*Constructors:* InertBody(void): (***default constructor***) executes Cartesian function atOrigin() for position, velocity, and initialPosition, sets bool \_at\_rest to TRUE, and assigns 0 to int \_n and \_DEFAULT\_TIME to int \_t.

InertBody(int a, int b): assigns (a, b) to Cartesian position, executes position.concordanceInt(), executes velocity.atOrigin(), executes Position() (sets initialPosition = position), sets bool \_at\_rest to TRUE, and assigns 0 to int \_n and \_DEFAULT\_TIME to int \_t.

InertBody(double aa, double bb): assigns (aa, bb) to Cartesian position, executes position.concordanceDouble(), executes velocity.atOrigin(), executes Position(), sets bool \_at\_rest to TRUE, and assigns 0 to int \_n and \_DEFAULT\_TIME to int \_t.

InertBody(Cartesian pos): assigns (pos.xx, pos.yy) to Cartesian position, executes position.concordanceDouble(), executes velocity.atOrigin(), executes Position(), sets bool \_at\_rest to TRUE, and assigns 0 to int \_n and \_DEFAULT\_TIME to int \_t.

InertBody(int a, int b, int time): assigns (a, b) to Cartesian position, executes position.concordanceInt(), executes velocity.atOrigin(), executes Position(), sets bool \_at\_rest to TRUE, and assigns 0 to int \_n and time to int \_t.

InertBody(double aa, double bb, int time): assigns (aa, bb) to Cartesian position, executes position.concordanceDouble(), executes velocity.atOrigin(), executes Position(), sets bool \_at\_rest to TRUE, and assigns 0 to int \_n and time to int \_t.

InertBody(Cartesian pos, int time): assigns (pos.xx, pos.yy) to Cartesian position, executes position.concordanceDouble(), executes velocity.atOrigin(), executes Position(), sets bool \_at\_rest to TRUE, and assigns 0 to int \_n and time to int \_t.

InertBody(int a, int b, int vx, int vy): assigns (a, b) to Cartesian position, executes position.concordanceInt(), assigns (vx, vy) to Cartesian velocity, executes velocity.concordanceInt(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, executes Position(), and assigns 0 to int \_n and \_DEFAULT\_TIME to int \_t.

InertBody(double aa, double bb, double vxx, double vyy): assigns (aa, bb) to Cartesian position, executes position.concordanceDouble(), assigns (vxx, vyy) to Cartesian velocity, executes velocity.concordanceDouble(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, executes Position(), and assigns 0 to int \_n and \_DEFAULT\_TIME to int \_t.

InertBody(Cartesian pos, Cartesian vel): assigns (pos.xx, pos.yy) to Cartesian position, executes position.concordanceDouble(), assigns (vel.xx, vel.yy) to Cartesian velocity, executes velocity.concordanceDouble(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, executes Position(), and assigns 0 to int \_n and \_DEFAULT\_TIME to int \_t.

InertBody(int a, int b, int vx, int vy, int time): assigns (a, b) to Cartesian position, executes position.concordanceInt(), assigns (vx, vy) to Cartesian velocity, executes velocity.concordanceInt(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, executes Position(), and assigns 0 to int \_n and time to int \_t.

InertBody(double aa, double bb, double vxx, double vyy, int time): assigns (aa, bb) to Cartesian position, executes position.concordanceDouble(), assigns (vxx, vyy) to Cartesian velocity, executes velocity.concordanceDouble(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, executes Position(), and assigns 0 to int \_n and time to int \_t.

InertBody(Cartesian pos, Cartesian vel, int time): assigns (pos.xx, pos.yy) to Cartesian position, executes position.concordanceDouble(), assigns (vel.xx, vel.yy) to Cartesian velocity, executes velocity.concordanceDouble(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, executes Position(), and assigns 0 to int \_n and time to int \_t.

InertBody(int a, int b, int vx, int vy, int ix, int iy): assigns (a, b) to Cartesian position, executes position.concordanceInt(), assigns (vx, vy) to Cartesian velocity, executes velocity.concordanceInt(), assigns (ix, iy) to Cartesian initialPosition, executes initialPosition.concordanceInt(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, and assigns 0 to int \_n and \_DEFAULT\_TIME to int \_t.

InertBody(double aa, double bb, double vxx, double vyy, double ixx, double iyy): assigns (aa, bb) to Cartesian position, executes position.concordanceDouble(), assigns (vxx, vyy) to Cartesian velocity, executes velocity.concordanceDouble(), assigns (ixx, iyy) to Cartesian initialPosition, executes initialPosition.concordanceDouble(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, and assigns 0 to int \_n and \_DEFAULT\_TIME to int \_t.

InertBody(Cartesian pos, Cartesian vel, Cartesian ipos): assigns (pos.xx, pos.yy) to Cartesian position, executes position.concordanceDouble(), assigns (vel.xx, vel.yy) to Cartesian velocity, executes velocity.concordanceDouble(), assigns (ipos.xx, ipos.yy) to Cartesian initialPosition, executes initialPosition.concordanceDouble(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, and assigns 0 to int \_n and \_DEFAULT\_TIME to int \_t.

InertBody(int a, int b, int vx, int vy, int ix, int iy, int time): assigns (a, b) to Cartesian position, executes position.concordanceInt(), assigns (vx, vy) to Cartesian velocity, executes velocity.concordanceInt(), assigns (ix, iy) to Cartesian initialPosition, executes initialPosition.concordanceInt(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, and assigns 0 to int \_n and time to int \_t.

InertBody(double aa, double bb, double vxx, double vyy, double ixx, double iyy, int time): assigns (aa, bb) to Cartesian position, executes position.concordanceDouble(), assigns (vxx, vyy) to Cartesian velocity, executes velocity.concordanceDouble(), assigns (ixx, iyy) to Cartesian initialPosition, executes initialPosition.concordanceDouble(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, and assigns 0 to int \_n and time to int \_t.

InertBody(Cartesian pos, Cartesian vel, Cartesian ipos, int time): assigns (pos.xx, pos.yy) to Cartesian position, executes position.concordanceDouble(), assigns (vel.xx, vel.yy) to Cartesian velocity, executes velocity.concordanceDouble(), assigns (ipos.xx, ipos.yy) to Cartesian initialPosition, executes initialPosition.concordanceDouble(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, and assigns 0 to int \_n and time to int \_t.

InertBody(const InertBody &bodyCopy): (***copy constructor***) assigns (bodyCopy.position.xx, bodyCopy.position.yy) to Cartesian position of current InertBody object, executes position.concordanceDouble(), assigns (bodyCopy.velocity.xx, bodyCopy.velocity.yy) to velocity of current InertBody object, executes velocity.concordanceDouble(), assigns (bodyCopy.initialPosition.xx, bodyCopy.initialPosition.yy) to Cartesian initialPosition, executes initialPosition.concordanceDouble(), creates Cartesian Zero.atOrigin(), sets bool \_at\_rest to TRUE if velocity and Zero are equal, resets it to FALSE otherwise, and assigns bodyCopy.\_n to int \_n and bodyCopy.\_t to int \_t.

*Operators:* ***Assignment*** (InertBody operator=(const InertBody &bodyAssignment)): Same as ***copy constructor***, but checks for self-assignment, for which case nothing happens, and uses bodyAssignment to copy from instead of bodyCopy, and at the end, the current object (\*this) is returned.

***Equality***(bool operator==(const InertBody &bodyEqual1, const InertBody &bodyEqual2)): assigns false to a new bool equality, compares Cartesian bodyEqual1.position and Cartesian bodyEqual2.position to see if they are equal (using class Cartesian’s equality operator). If true, Cartesian bodyEqual1.velocity and Cartesian bodyEqual2.velocity are compared to see if they are equal. If true, Cartesian bodyEqual1.initialPosition and Cartesian bodyEqual2.initialPosition are compared to see if they are equal. If true, true is assigned to equality. Otherwise, if any of these tests fail, false remains assigned to equality. Finally, equality is returned.

***Inequality***(bool operator!=(const InertBody &bodyUnequal1, const InertBody &bodyUnequal2)): assigns false to a new bool inequality, and compares Cartesian bodyUnequal1.position and Cartesian bodyUnequal2.position to see if they are unequal. If so, true is assigned to inequality. Otherwise, Cartesian bodyUnequal1.velocity is compared to Cartesian bodyUnequal2.velocity to see if they are unequal. If so, true is assigned to inequality. Otherwise, Cartesian bodyUnequal1.initialPosition is compared to Cartesian bodyUnequal2.initialPosition to see if they are unequal. If so, true is assigned to inequality. Otherwise, if all these tests fail, false remains assigned to inequality. Finally, inequality is returned.

*Destructor:* ~InertBody(void): deletes Cartesian position, Cartesian velocity, and Cartesian initialPosition before rest of current object is destroyed.

*Functions that read the position of the current object:*

int X(void): returns int position.x.

int Y(void): returns int position.y.

double XX(void): returns double position.xx.

double YY(void): returns double position.yy.

Cartesian Position(void): returns Cartesian position.

*Functions that assign the position of the current object:*

void X(int a): assigns a to int position.x and executes position.concordanceInt().

void Y(int b): assigns b to int position.y and executes position.concordanceInt().

void X\_Y(int a, int b): assigns a to int position.x, assigns b to int position.y, and executes position.concordanceInt().

void XX(double aa): assigns aa to double position.xx and executes position.concordanceDouble().

void YY(double bb): assigns bb to double position.yy and executes position.concordanceDouble().

void XX\_YY(double aa, double bb): assigns aa to double position.xx, assigns bb to double position.yy, and executes position.concordanceDouble().

void Position(Cartesian pos): assigns double pos.xx to double position.xx, assigns double pos.yy to double position.yy, and executes position.concordanceDouble().

void Position(void): assigns values of Cartesian position object to Cartesian initialPosition object by assigning double position.xx to double initialPosition.xx, assigning double position.yy to double initialPosition.yy, and executing initialPosition.concordanceDouble().

*Functions that read the velocity of the current object:*

int VX(void): returns int velocity.x.

int VY(void): returns int velocity.y.

double VXX(void): returns double velocity.xx.

double VYY(void): returns double velocity.yy.

Cartesian Velocity(void): returns Cartesian velocity

bool AtRest(void): returns bool \_at\_rest

*Functions that assign the velocity of the current object:*

void VX(int vx): assigns vx to int velocity.x and executes velocity.concordanceInt().

void VY(int vy): assigns vy to int velocity.y and executes velocity.concordanceInt().

void VX\_VY(int vx, int vy): assigns vx to int velocity.x, assigns vy to int velocity.y, and executes velocity.concordanceInt().

void VXX(double vxx): assigns vxx to double velocity.xx and executes velocity.concordanceDouble().

void VYY(double vyy): assigns vyy to double position.yy and executes position.concordanceDouble().

void VXX\_VYY(double vxx, double vyy): assigns vxx to double velocity.xx, assigns bb to double velocity.yy, and executes velocity.concordanceDouble().

void Velocity(Cartesian vel): assigns double vel.xx to double velocity.xx, assigns double vel.yy to double velocity.yy, and executes velocity.concordanceDouble().

*Functions that read the initial position of the current object:*

int initX(void): returns int initialPosition.x.

int initY(void): returns int initialPosition.y.

double initXX(void): returns double initialPosition.xx.

double initYY(void): returns double initialPosition.yy.

Cartesian initPosition(void): returns Cartesian initialPosition.

*Functions that assign the initial position of the current object:*

void initX(int ix): assigns ix to int initialPosition.x and executes initialPosition.concordanceInt().

void initY(int iy): assigns iy to int initialPosition.y and executes initialPosition.concordanceInt().

void initX\_Y(int ix, int iy): assigns ix to int initialPosition.x, assigns iy to int initialPosition.y, and executes initialPosition.concordanceInt().

void initXX(double ixx): assigns ixx to double initialPosition.xx and executes initialPosition.concordanceDouble().

void initYY(double iyy): assigns iyy to double initialPosition.yy and executes initialPosition.concordanceDouble().

void initXX\_YY(double ixx, double iyy): assigns ixx to double initialPosition.xx, assigns iyy to double initialPosition.yy, and executes initialPosition.concordanceDouble().

void initPosition(Cartesian ipos): assigns double ipos.xx to double initialPosition.xx, assigns double ipos.yy to double initialPosition.yy, and executes initialPosition.concordanceDouble().

void initPosition(void): assigns values of Cartesian initialPosition object to Cartesian position object by assigning double initialPosition.xx to double position.xx, assigning double initialPosition.yy to double position.yy, and executing position.concordanceDouble().

void posCurInitJux(void): juxtaposes the values of Cartesian position and Cartesian initialPosition by creating new Cartesian jux and executing jux.atOrigin(), assigning double position.xx to double jux.xx and double position.yy to jux.yy, assigning double initialPosition.xx to position.xx and double iniialPosition.yy to position.yy, executing position.concordanceDouble(), assigning jux.xx to initialPosition.xx and jux.yy to initialPosition.yy, executing initialPosition.concodanceDouble(), and finally deleting jux.

*Functions that assign the next position in the path of the current object and then return it:*

int nextX(void): increments int \_n, if position.concordance() returns false, executes position.concordanceInt(), then if velocity.concordance() returns false, executes velocity.concordanceInt(), creates a new double tt and assigns (double) \_n / (double) int \_t to it, assigns double initialPosition.xx + (double velocity.xx \* tt) to double position.xx, executes position.concordanceDouble(), and returns int position.x.

int nextY(void): increments int \_n, if position.concordance() returns false, executes position.concordanceInt(), then if velocity.concordance() returns false, executes velocity.concordanceInt(), creates a new double tt and assigns (double) \_n / (double) int \_t to it, assigns double initialPosition.yy + (double velocity.yy \* tt) to double position.yy, executes position.concordanceDouble(), and returns int position.y.

double nextXX(void): increments int \_n, creates a new double tt and assigns (double) \_n / (double) int \_t to it, assigns double initialPosition.xx + (double velocity.xx \* tt) to double position.xx, executes position.concordanceDouble(), and returns position.xx.

double nextYY(void): increments int \_n, creates a new double tt and assigns (double) \_n / (double) int \_t to it, assigns double initialPosition.yy + (double velocity.yy \* tt) to double position.yy, executes position.concordanceDouble(), and returns position.yy.

Cartesian nextPosition(void): increments int \_n, creates a new double tt and assigns (double) \_n / (double) int \_t to it, assigns double initialPosition.xx + (double velocity.xx \* tt) to double position.xx, assigns double initialPosition.yy + (double velocity.yy \* tt) to double position.yy, executes position.concordanceDouble(), and returns Cartesian position.

*Functions that assign the previous position in the path of the current object and then return it:*

int prevX(void): decrements int \_n, if position.concordance() returns false, executes position.concordanceInt(), then if velocity.concordance() returns false, executes velocity.concordanceInt(), creates a new double tt and assigns (double) \_n / (double) int \_t to it, assigns double initialPosition.xx + (double velocity.xx \* tt) to double position.xx, executes position.concordanceDouble(), and returns int position.x.

int prevY(void): decrements int \_n, if position.concordance() returns false, executes position.concordanceInt(), then if velocity.concordance() returns false, executes velocity.concordanceInt(), creates a new double tt and assigns (double) \_n / (double) int \_t to it, assigns double initialPosition.yy + (double velocity.yy \* tt) to double position.yy, executes position.concordanceDouble(), and returns int position.y.

double prevXX(void): decrements int \_n, creates a new double tt and assigns (double) \_n / (double) int \_t to it, assigns double initialPosition.xx + (double velocity.xx \* tt) to double position.xx, executes position.concordanceDouble(), and returns position.xx.

double prevYY(void): decrements int \_n, creates a new double tt and assigns (double) \_n / (double) int \_t to it, assigns double initialPosition.yy + (double velocity.yy \* tt) to double position.yy, executes position.concordanceDouble(), and returns position.yy.

Cartesian prevPosition(void): decrements int \_n, creates a new double tt and assigns (double) \_n / (double) int \_t to it, assigns double initialPosition.xx + (double velocity.xx \* tt) to double position.xx, assigns double initialPosition.yy + (double velocity.yy \* tt) to double position.yy, executes position.concordanceDouble(), and returns Cartesian position.

*Functions that return and assign protected members of the current object:*

int TimeSlice(void): return int \_t.

int NumberOfScices(void): return int \_n.

void TimeSlice(int time): assign time to int \_t.

void NumberOfSlices(int num): assign num to int \_n.

class GravitationalBody

The class GravitationalBody is a derived class of class InertBody and has all the members from it (public: Cartesian position, Cartesian velocity, Cartesian initialPosition; and protected: bool \_at\_rest, int \_n, and int \_t) plus its own members (public: Cartesian initialVelocity; and protected: int \_ground, double \_gground, long int \_accgrav, double \_aaccgrav, int \_cor100, double \_ccor, bool \_groundOn, and bool \_gravOn). Whereas objects of class InertBody do not have any acceleration applied to them, objects of class GravitationalBody do. Although this acceleration can be any number directly encoded into long int \_accgrav or double \_aaccgrav, or by using the functions \_G(long int) or \_G(double), typically it would be one of the predefined constants below for terrestrial gravitational acceleration, or by using one of the functions that assigns them.

*Constants defined by precompiler:*

GSI is 9.80665 (Standard gravity on Earth in SI units = 9.80665 m/sec²)

GCGS is 980.665 (Standard gravity on Earth in CGS units = 980.665 cm/sec²)

GMS2 is 9 806 650 (Standard gravity on Earth in microns/sec² = 9,806,650 microns/sec²)

GUS is 32.1740 (Standard gravity on Earth in US units = 32.1740 ft/sec²)

GIS2 is 386.088 (Standard gravity on Earth in inches/sec² = 386.088 in/sec²)

G1000IS2 is 386 088 (Standard gravity on Earth in 1000ths of inches/sec² = 386,088 in/1000sec²)

*Variable Member:* Cartesian initialVelocity.

**Protected** *Variable Members:*

int \_ground, double \_gground (horizontal y/yy-level ground)

long int \_accgrav, double \_aaccgrav (acceleration of gravity)

int \_cor100 (percentage of coefficient of restitution—bounceback from ground)

double \_ccor (same as int \_cor100, but expressed as a pure decimal)

bool \_groundOn (switches horizontal ground on or off)

bool \_gravOn (switches the gravity on or off)

**Inherited** *Variable Members:* Cartesian objects: position, velocity, initialPosition (inherited from class InertBody).

**Inherited Protected** *Variable Members:* bool \_at\_rest, ints: \_n, \_t (inherited from class InertBody).

*Constructors:*

// constructors

GravitationalBody(void);

GravitationalBody(int a, int b);

GravitationalBody(double aa, double bb);

GravitationalBody(Cartesian pos);

GravitationalBody(int a, int b, int vx, int vy);

GravitationalBody(double aa, double bb, double vxx, double vyy);

GravitationalBody(Cartesian pos, Cartesian vel);

GravitationalBody(int a, int b, int vx, int vy, int ix, int iy);

GravitationalBody(double aa, double bb, double vxx, double vyy, double ixx, double iyy);

GravitationalBody(Cartesian pos, Cartesian vel, Cartesian ipos);

GravitationalBody(int a, int b, int vx, int vy, int ix, int iy, int ivx, int ivy);

GravitationalBody(double aa, double bb, double vxx, double vyy, double ixx, double iyy, double ivxx, double ivyy);

GravitationalBody(Cartesian pos, Cartesian vel, Cartesian ipos, Cartesian ivel);

GravitationalBody(int a, int b, int vx, int vy, int ix, int iy, int ivx, int ivy, int grnd);

GravitationalBody(double aa, double bb, double vxx, double vyy, double ixx, double iyy, double ivxx, double ivyy, double ggrnd);

GravitationalBody(Cartesian pos, Cartesian vel, Cartesian ipos, Cartesian ivel, double ggrnd);

GravitationalBody(int a, int b, int vx, int vy, int ix, int iy, int ivx, int ivy, int grnd, int agv);

GravitationalBody(double aa, double bb, double vxx, double vyy, double ixx, double iyy, double ivxx, double ivyy, double ggrnd, double aagv);

GravitationalBody(Cartesian pos, Cartesian vel, Cartesian ipos, Cartesian ivel, double ggrnd, double aagv);

GravitationalBody(int a, int b, int vx, int vy, int ix, int iy, int ivx, int ivy, int grnd, int agv, int epsln100);

GravitationalBody(double aa, double bb, double vxx, double vyy, double ixx, double iyy, double ivxx, double ivyy, double ggrnd, double aagv, double epsilon);

GravitationalBody(Cartesian pos, Cartesian vel, Cartesian ipos, Cartesian ivel, double ggrnd, double aagv, double epsilon);

GravitationalBody(const GravitationBody &gravbodCopy);

GravitationalBody operator=(const GravitationalBody &gravbodAssignment);

class TerrestrialGravitation

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• d) Convey the object code by offering access from a designated place (gratis or for a charge), and offer equivalent access to the Corresponding Source in the same way through the same place at no further charge. You need not require recipients to copy the Corresponding Source along with the object code. If the place to copy the object code is a network server, the Corresponding Source may be on a different server (operated by you or a third party) that supports equivalent copying facilities, provided you maintain clear directions next to the object code saying where to find the Corresponding Source. Regardless of what server hosts the Corresponding Source, you remain obligated to ensure that it is available for as long as needed to satisfy these requirements.

• e) Convey the object code using peer-to-peer transmission, provided you inform other peers where the object code and Corresponding Source of the work are being offered to the general public at no charge under subsection 6d.

A separable portion of the object code, whose source code is excluded from the Corresponding Source as a System Library, need not be included in conveying the object code work.

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